Iteration Logic

A Modal-Like Logic/Calculus/Algebra for Reasoning about Fixed Points of Continuous Maps

Kevin Batz **Benjamin Lucien Kaminski** Lucas Kehrer
Henning Urbat Todd Schmid

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Talk at the 69$^{\text{TH}}$ MEETING *of the* IFIP WORKING GROUP *on* PROGRAMMING METHODOLOGY

May 19, 2025, Athens, Greece

What I Usually Do

Weakest Preconditions

$/\!\!/\!\!/ \; [\boldsymbol{x} > \mathbf{5} \;\wedge\; \boldsymbol{x} \text{ odd}] \;\vee\; [\boldsymbol{x} \leq \mathbf{5} \;\wedge\; \textbf{true}]$

$\texttt{if}\ (\, x > 5 \,)\ \{$

$\quad /\!\!/\!\!/ \; [\boldsymbol{x} + \mathbf{1} \text{ even}]$

$\quad x \mathrel{:=} x + 1$

$\quad /\!\!/\!\!/ \; [\boldsymbol{x} \text{ even}]$

$\}\,\texttt{else}\,\{$

$\quad /\!\!/\!\!/ \; [\mathbf{2} \text{ even}]$

$\quad x \mathrel{:=} 2$

$\quad /\!\!/\!\!/ \; [\boldsymbol{x} \text{ even}]$

$\}$

$/\!\!/\!\!/ \; [\boldsymbol{x} \text{ even}]$

Weakest Preexpectations

$$/\!\!/\!\!/ \;\; \tfrac{1}{2} \cdot (x + 1) \;\; + \;\; \tfrac{1}{2} \cdot 2$$

$$\{$$

$$\quad /\!\!/\!\!/ \;\; x + 1$$

$$\quad x := x + 1$$

$$\quad /\!\!/\!\!/ \;\; x$$

$$\} \; \tfrac{1}{2} \; \{$$

$$\quad /\!\!/\!\!/ \;\; 2$$

$$\quad x := 2$$

$$\quad /\!\!/\!\!/ \;\; x$$

$$\}$$

$$/\!\!/\!\!/ \;\; x$$

What about loops?

$$/\!\!/\!\!/\ \ \boldsymbol{x+1}$$

$$\texttt{while}\left(\tfrac{1}{2}\right)\{\, x \mathrel{\texttt{:=}} x+1 \,\}$$

$$/\!\!/\!\!/\ \ \boldsymbol{x}$$

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The Setting

FIGURE 4. Brouwer's fixed point theorem. The black dot stays place after a rotation and a swirl.

[0]Bocklandt. *Reflections in a Cup of Coffee*. 2016

The Liquid - A Complete Lattice

- A complete lattice L where every countable subset has supremum and infimum
- partial order $\leq$
  - reflexive: $a \leq a$
  - transitive: $a \leq b \leq c$ implies $a \leq c$
  - antisymmetric: $a \leq b$ and $b \leq a$ implies $a = b$
- bottom element $\bot$, top element $\top$
- binary join $\curlywedge$, binary meet $\curlyvee$

The Stirring Spoon - A Monotonic Map (ω-continuous)

- A monotonic endomap $F\colon L \to L$.
- Possibly, F is *continuous*, meaning

ω-continuous: For every ascending chain $\{a_0 \leq a_1 \leq a_2 \leq \ldots\} \subseteq L$,

$$F\left(\sup_k a_k\right) = \sup_k F(a_k)$$

ω-cocontinuous: For every descending chain $\{a_0 \geq a_1 \geq a_2 \geq \ldots\} \subseteq L$,

$$F\left(\inf_k a_k\right) = \inf_k F(a_k)$$

The Stirring Spoon - A Monotonic Map (countably continuous)

- A monotonic endomap $F\colon L \to L$.
- Possibly, F is *continuous*, meaning

countably continuous: For every subset $\{a_0, a_1, a_2, \ldots\} \subseteq L$,

$$F\left(\sup_k a_k\right) \quad = \quad \sup_k F(a_k)$$

countably cocontinuous: For every subset $\{a_0, a_1, a_2, \ldots\} \subseteq L$,

$$F\left(\inf_k a_k\right) \quad = \quad \inf_k F(a_k)$$

The Molecules - The Lattice Elements

- $a \in L$
- Fixed points: $F(a) = a$

What happens to individual a's, if we stir them through F?

Fixed Point (Iteration) Theorems

A Very Well-Known Fixed Point Iteration

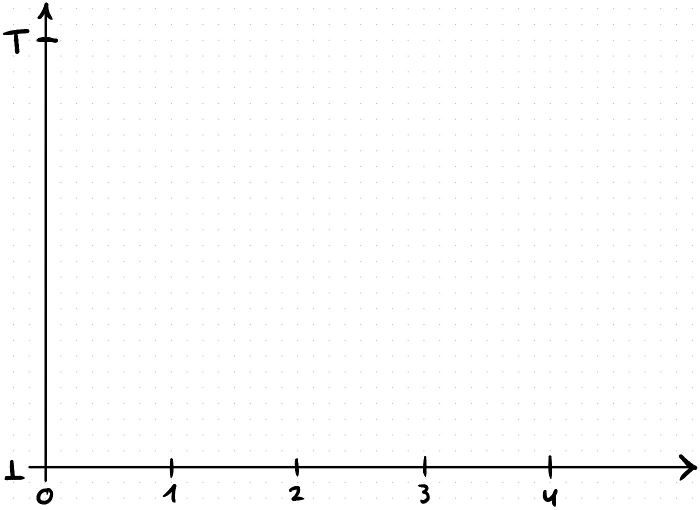
Kleene Fixed Point Theorem

If F is ω-continuous then F has a least fixed point lfp F and moreover

$$\sup_k F^k(\bot) \;=\; \mathsf{lfp}\ F$$

If F is ω-cocontinuous then F has a greatest fixed point gfp F and moreover

$$\inf_k F^k(\top) \;=\; \mathsf{gfp}\ F$$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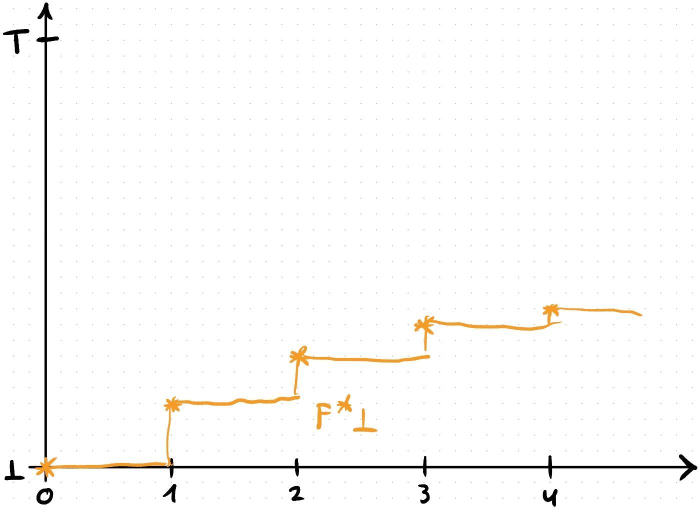
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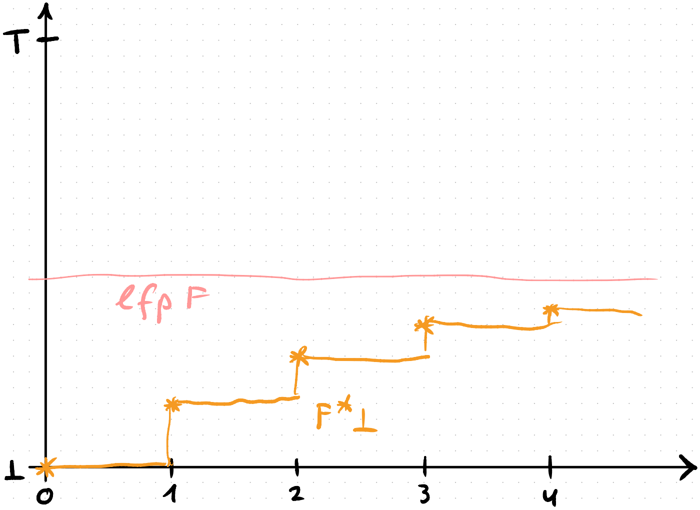
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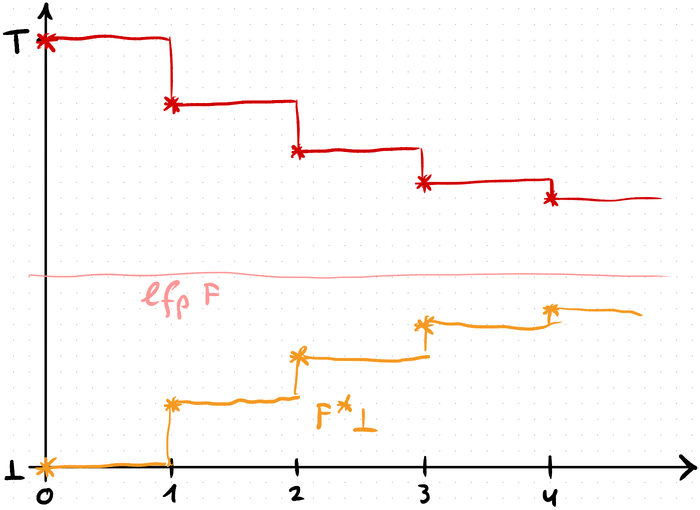
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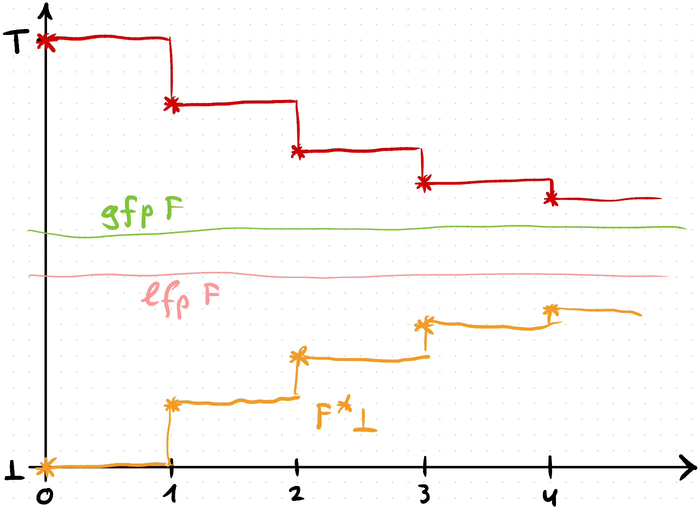
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A Slightly Lesser Known Fixed Point Iteration

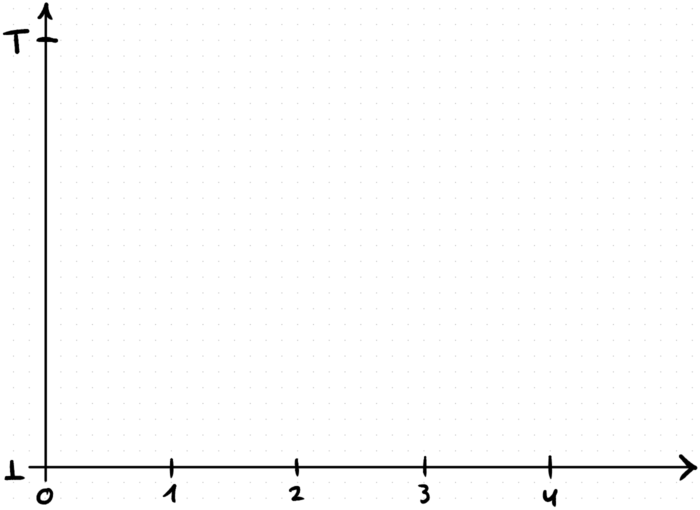
Tarski-Kantorovich Principle

Let F be ω-continuous and $a \in L$, such that $a \leq F(a)$. Then

$$\sup_k F^k(a) \quad \text{is the least fixed point above } a.$$

Let F be ω-cocontinuous and $a \in L$, such that $F(a) \leq a$. Then

$$\inf_k F^k(a) \quad \text{is the greatest fixed point below } a.$$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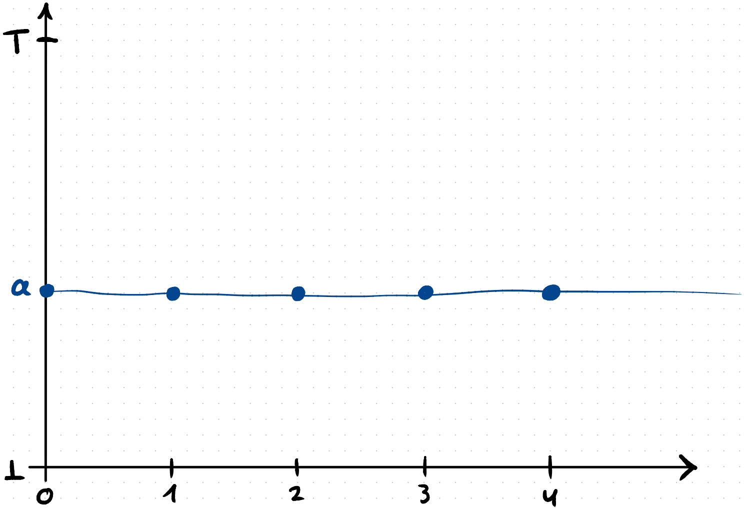
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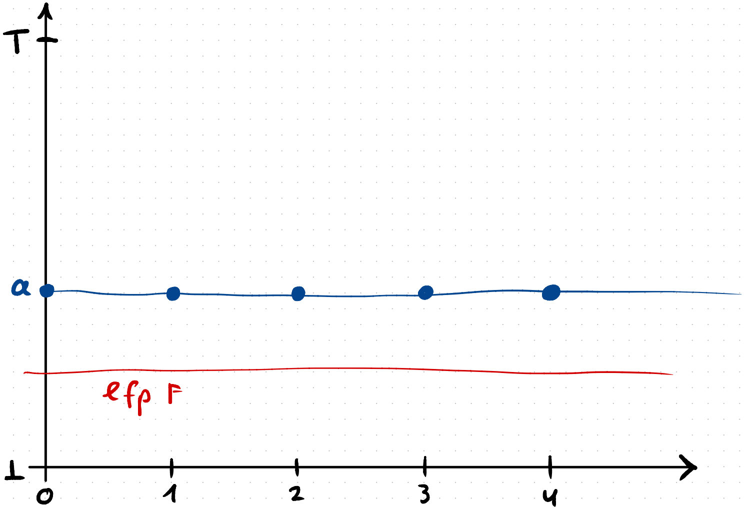
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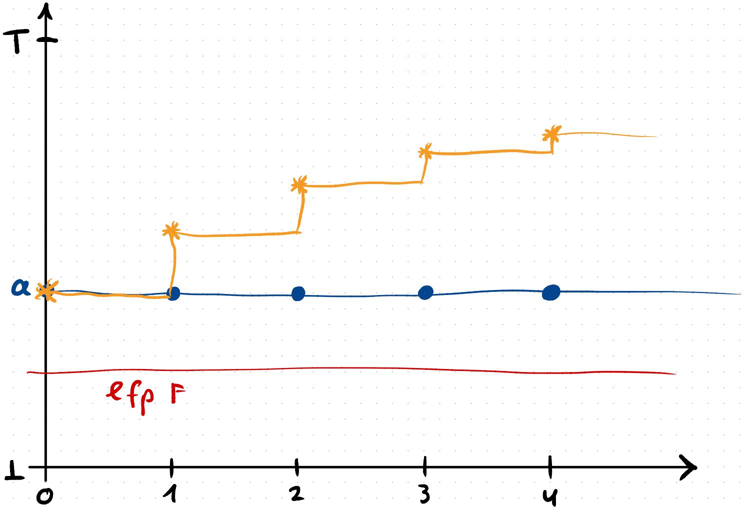
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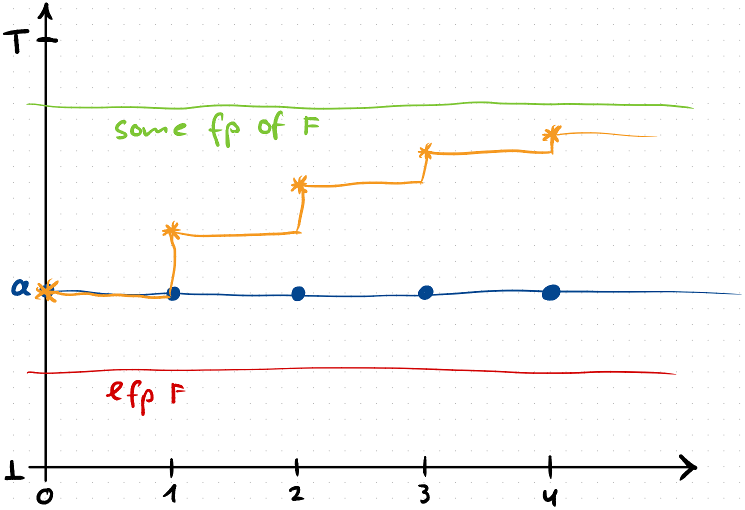
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An Essentially Unknown Fixed Point Iteration

Let <u>neither</u> $\boldsymbol{a \preceq F(a)}$ **<u>nor</u>** $\boldsymbol{F(a) \preceq a}$ **.**

Then fixed point iteration a, $F(a)$, $F^2(a)$, $F^3(a)$, $\ldots$ is not a chain and

$$\sup_k F^k(a)$$ is *not* a fixed point

We just went modal!

Is this iteration really unknown?

Theorem [essentially Olszewski 2021]

Let F be ω-continuous and ω-cocontinuous. Then

$$\sup_{m} F^m\left(\sup_{k} \inf_{k \leq j} F^j(a)\right) \quad \text{is } \textit{some} \text{ fixed point}$$

and

$$\inf_{m} F^m\left(\inf_{k} \sup_{k \leq j} F^j(a)\right) \quad \text{is } \textit{some} \text{ (larger) fixed point}$$

An Actually Unknown Fixed Point Iteration

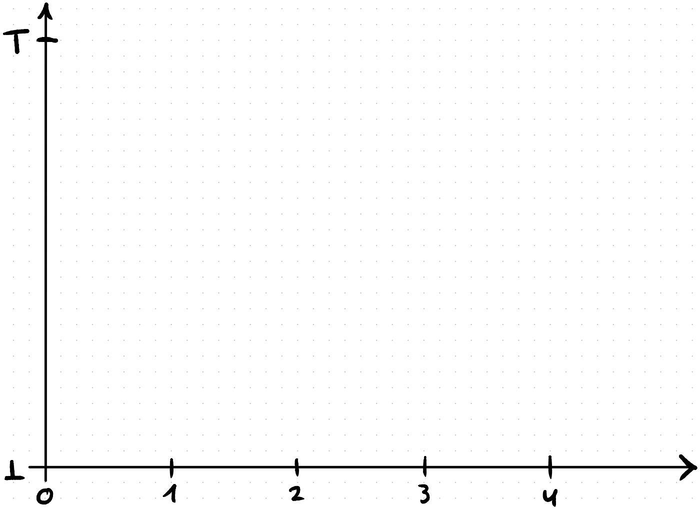
A New Theorem

Let F be ω-continuous and countably cocontinuous. Then

$$\sup_{k} \inf_{k \leq j} F^{j}(a) \quad \text{is } \textit{some} \text{ fixed point}$$

and if F is countably continuous and ω-cocontinuous.

$$\inf_{k} \sup_{k \leq j} F^{j}(a) \quad \text{is } \textit{some} \text{ (larger) fixed point}$$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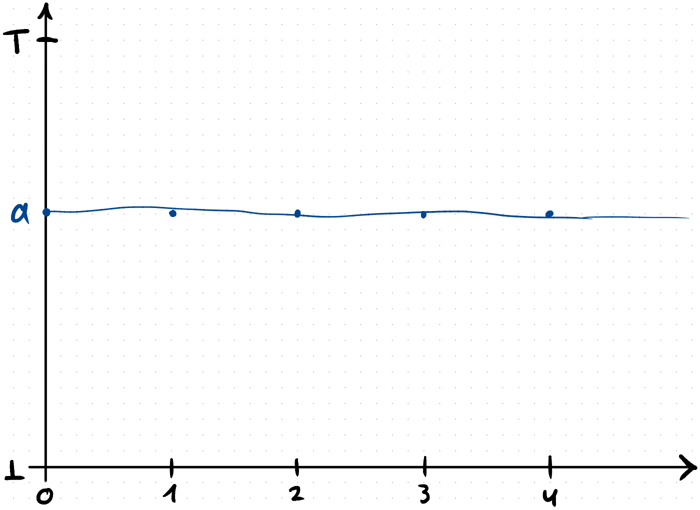
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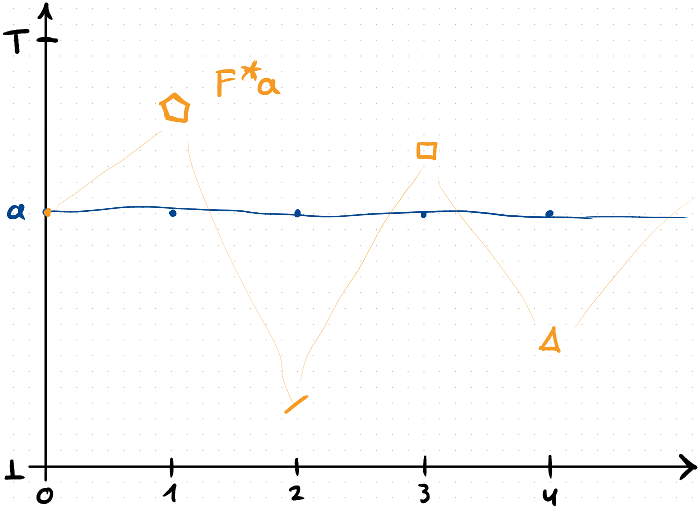
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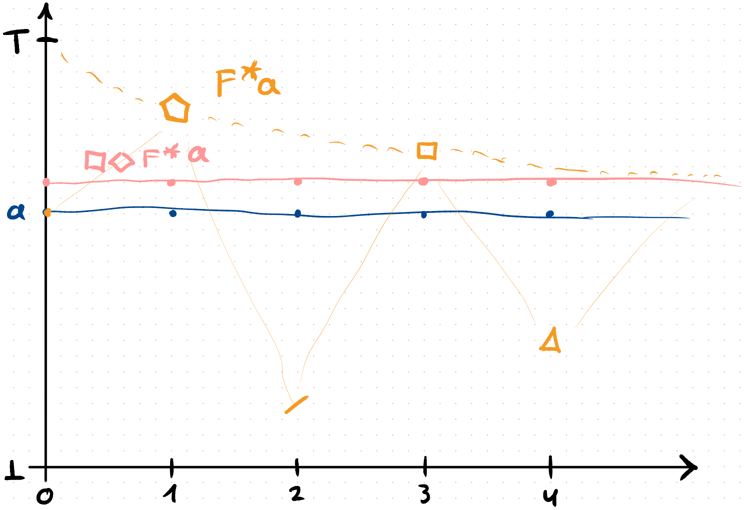
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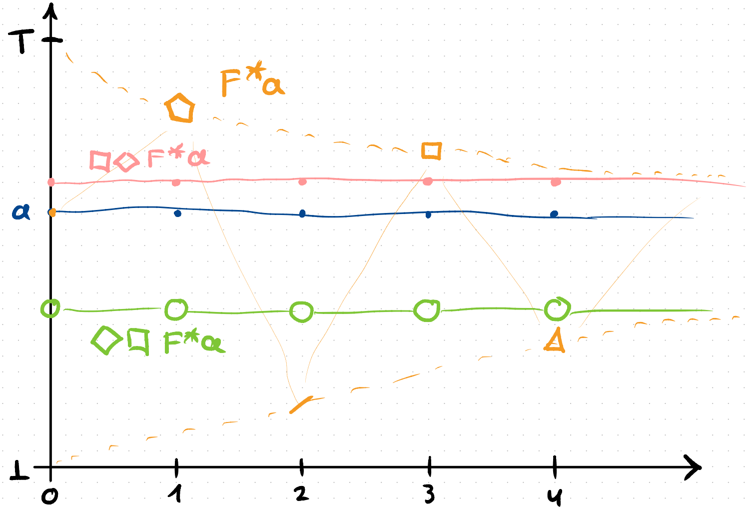
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Quick Poll

1. I knew about Olszewski's theorem
2. I think this is folklore knowledge
3. I know similar Olszewski-type theorems (perhaps in other arease), namely...
4. I find Olszewski's theorem intuitive and/or completely obvious

Iteration Logic

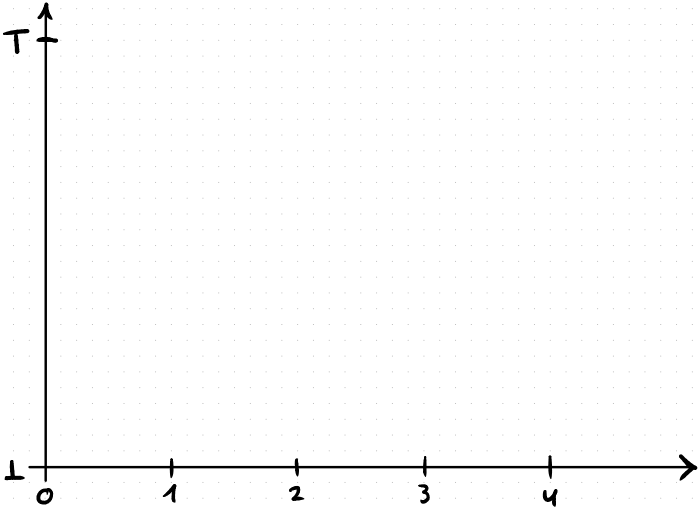
Principles of Iteration Logic (IL)

- First class citizens of IL are sequences $(a_n)_{n \in \mathbb{N}}$, where $a_n \in L$
- Fixed point iterations should be "representable" sequences
- Limit behavior of sequences should be "representable"
- Fixed point theorems should be *provable*

Syntax and Semantics and Intuition

Syntax and Semantics of Iteration Logic (Atoms)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$\bot$	$\bot$	constant sequence of bottom elements $\bot$
$\top$	$\top$	constant sequence of top elements $\bot$
x	x_n	variable representing an arbitrary sequence $(x_n)_{n\in\mathbb{N}}$ of lattice elements of L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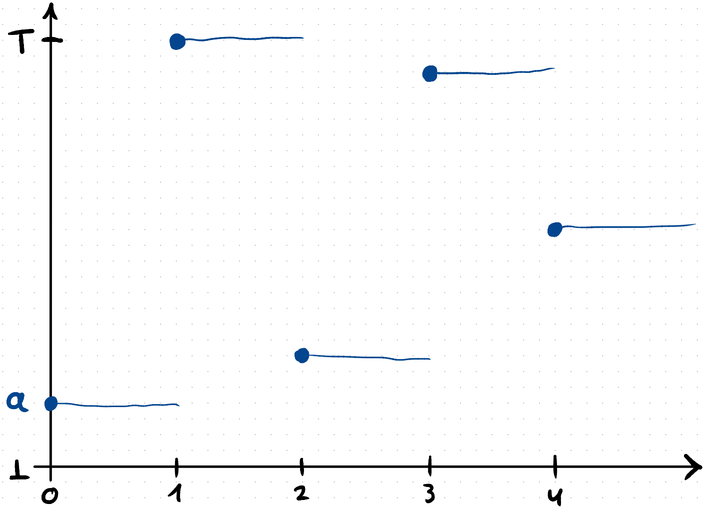
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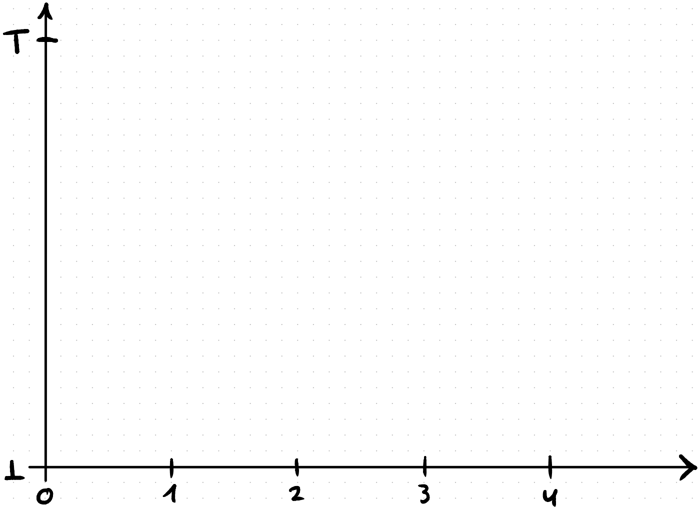
Syntax and Semantics of Iteration Logic (Binary Connectives)

Formula Symbol	Semantics	Remarks
$\boldsymbol{a}$	$[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	
$a \curlyvee b$	$[\![a]\!]_n \;\curlyvee\; [\![b]\!]_n$	point-wise meet of sequences a and b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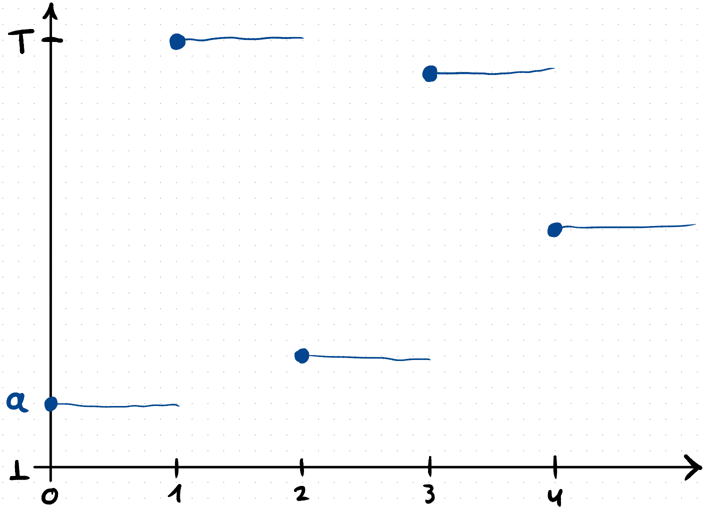
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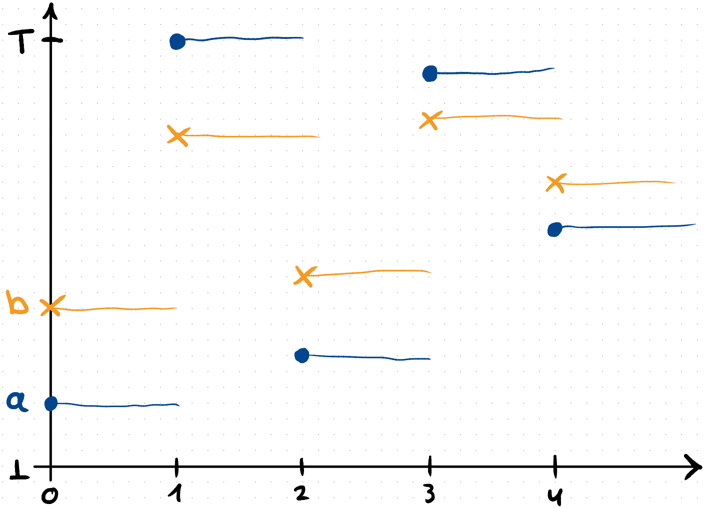
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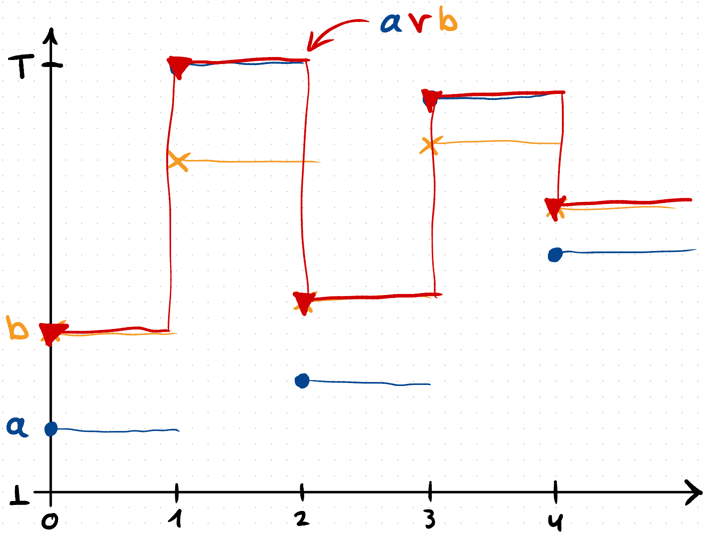
a ∨ b
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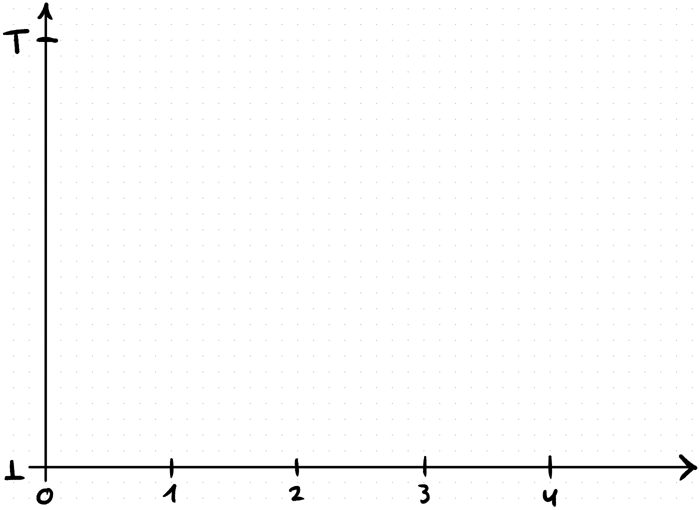
Syntax and Semantics of Iteration Logic (Binary Connectives)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$a \curlyvee b$	$[\![a]\!]_n \curlyvee [\![b]\!]_n$	point-wise meet of sequences a and b
$a \curlywedge b$	$[\![a]\!]_n \curlywedge [\![b]\!]_n$	point-wise join of sequences a and b

Syntax and Semantics of Iteration Logic (The Modalities)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$\Diamond\, a$	$\sup_{n \leq k} [\![a]\!]_k$	n-th element is the supremum over all elements with index $\geq n$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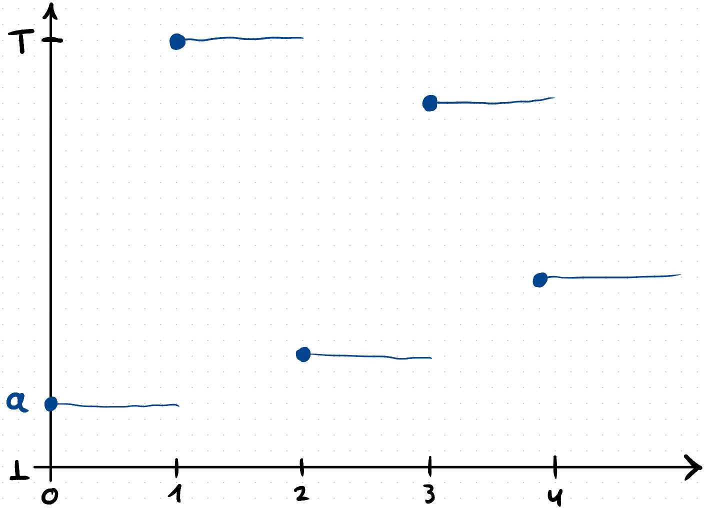
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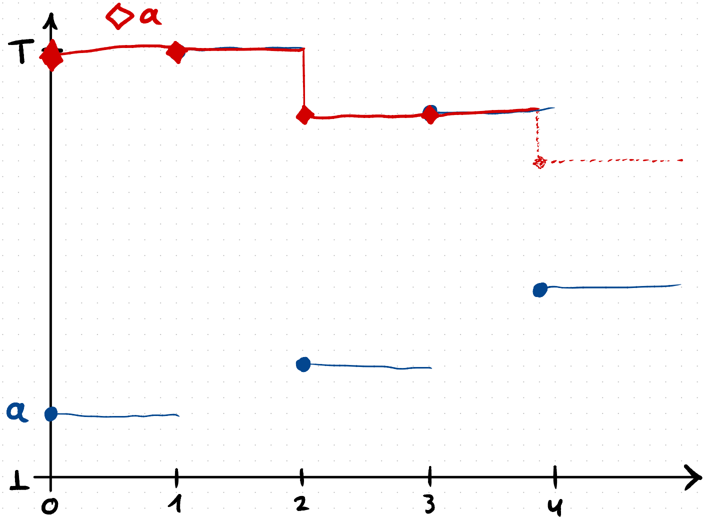
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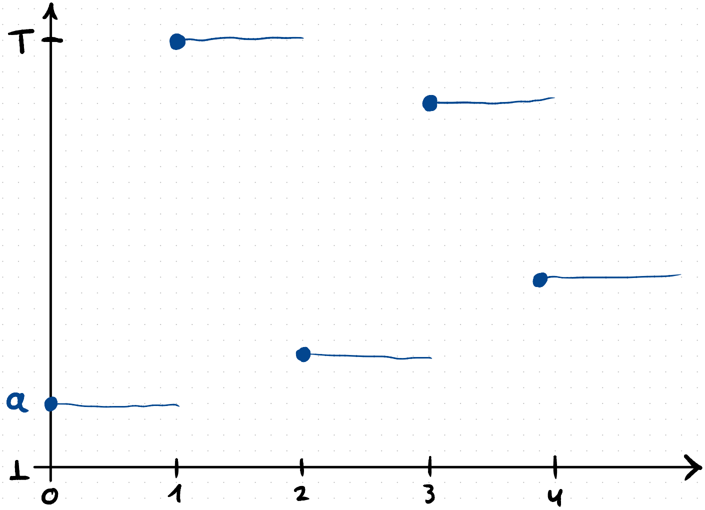
Syntax and Semantics of Iteration Logic (The Modalities)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$\Diamond\, a$	$\sup\limits_{n \leq k} [\![a]\!]_k$	n-th element is the supremum over all elements with index $\geq n$
$\Box\, a$	$\inf\limits_{n \leq k} [\![a]\!]_k$	n-th element is the infimum over all elements with index $\geq n$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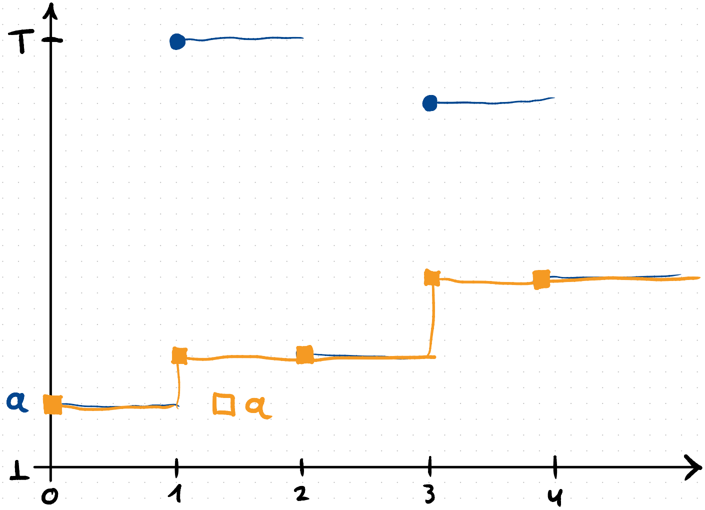
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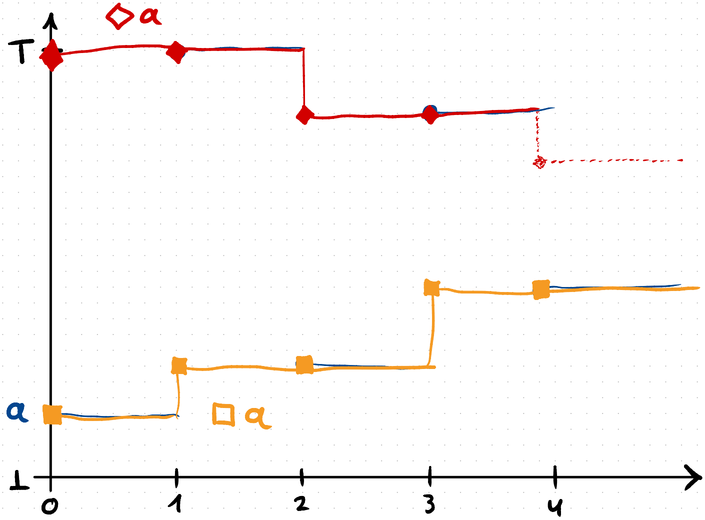
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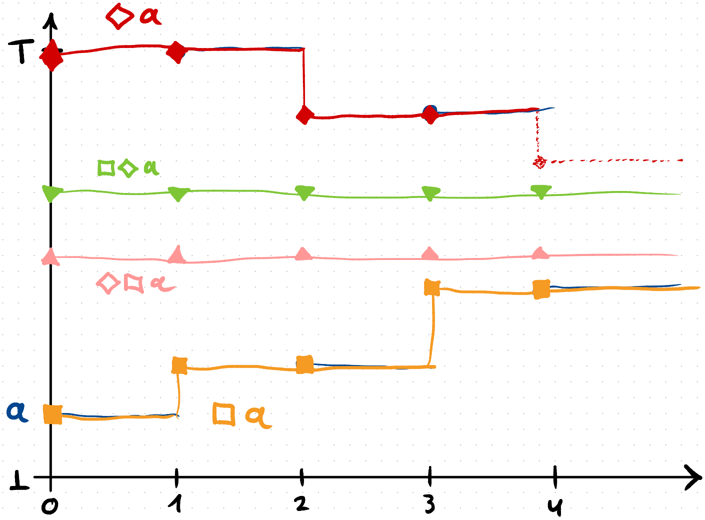
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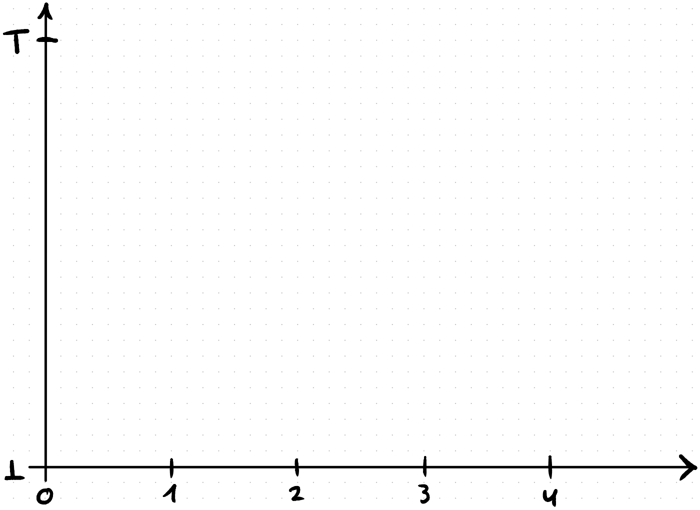
Syntax and Semantics of Iteration Logic (The Modalities)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$\Diamond\, a$	$\sup_{n \leq k}\, [\![a]\!]_k$	n-th element is the supremum over all elements with index $\geq n$
$\Box\, a$	$\inf_{n \leq k}\, [\![a]\!]_k$	n-th element is the infimum over all elements with index $\geq n$
$\bigcirc\, a$	$[\![a]\!]_{n+1}$	drops 0-th element and left-shifts all other elements by 1 index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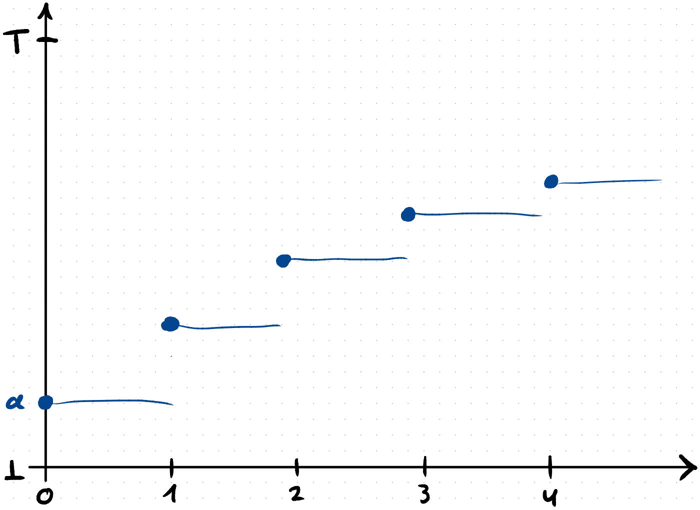
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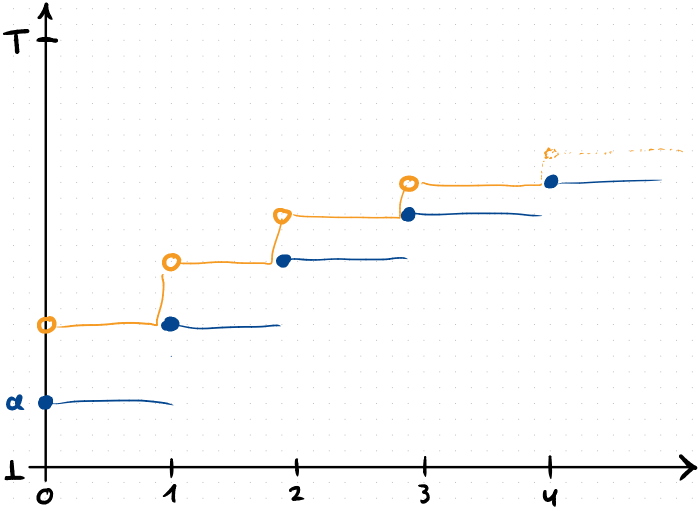
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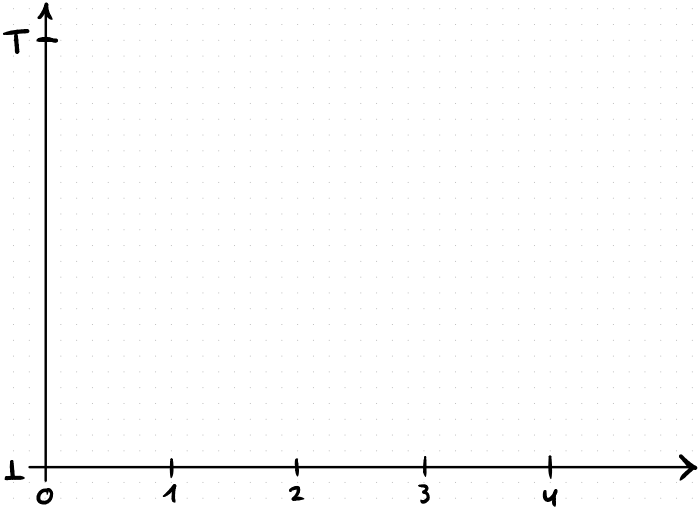
Syntax and Semantics of Iteration Logic (The Modalities)

Formula Symbol $\boldsymbol{a}$	**Semantics** $[\![\boldsymbol{a}]\!]_{\boldsymbol{n}} \in L$	**Remarks**
$\Diamond\, a$	$\sup\limits_{n \leq k}\ [\![a]\!]_k$	n-th element is the supremum over all elements with index $\geq n$
$\Box\, a$	$\inf\limits_{n \leq k}\ [\![a]\!]_k$	n-th element is the infimum over all elements with index $\geq n$
$\bigcirc\, a$	$[\![a]\!]_{n+1}$	drops 0-th element and left-shifts all other elements by 1 index
$F\, a$	$F\big([\![a]\!]_n\big)$	applies F element-wise to entire sequence
F^*a	$F^n\big([\![a]\!]_n\big)$	n-th element is the n-fold iteration of F on the n-th element of a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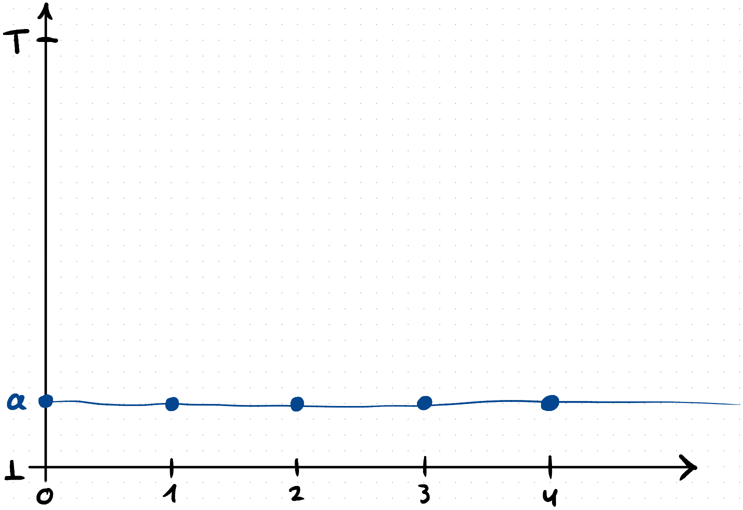
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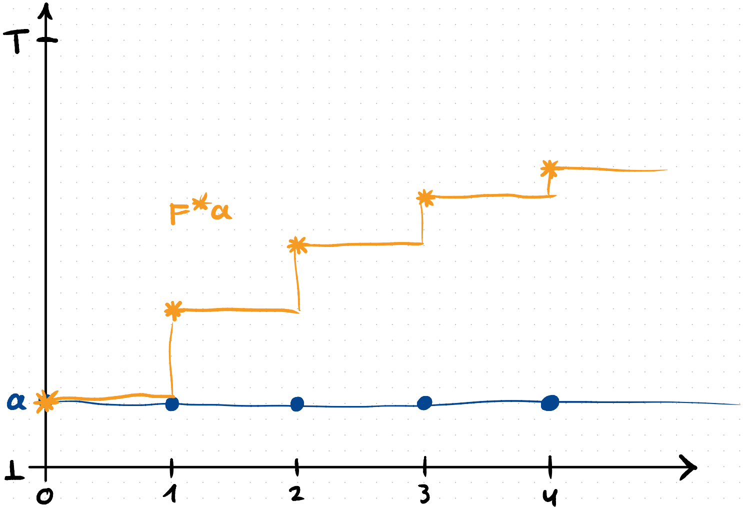
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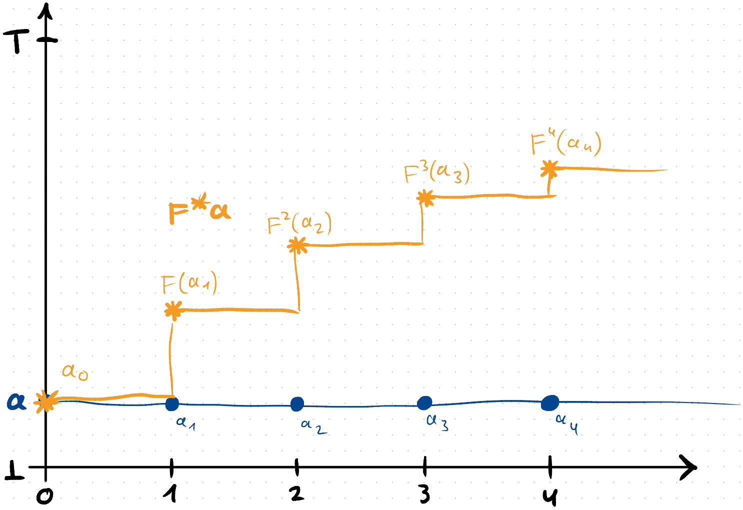
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Judgements

Judgements

A comparant (a *judgement*) in iteration logic is of the form:

$$a \preceq b$$

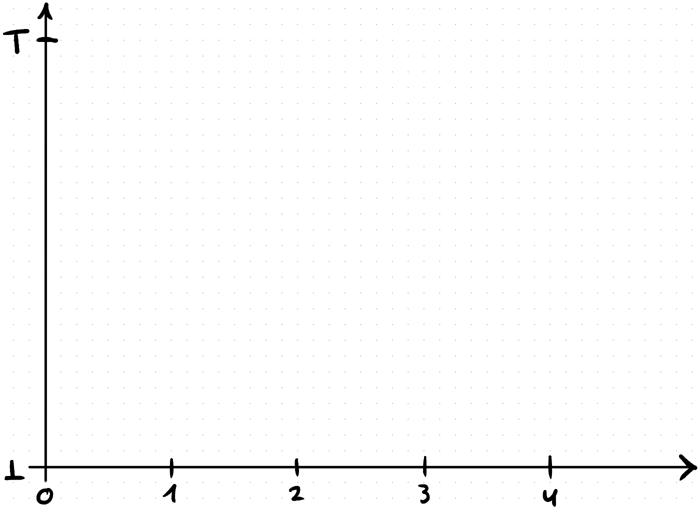
A comparant $a \preceq b$ is valid iff for all $n \in \mathbb{N}$:[1]

$$[\![a]\!]_n \preceq [\![b]\!]_n$$

[1] $a \equiv b$ is a shorthand for validity of $[\![a]\!]_n \preceq [\![b]\!]_n$ and $[\![b]\!]_n \preceq [\![a]\!]_n$

Interesting Judgements

$$a \;\leq\; \bigcirc\, a \qquad a \text{ is an ascending chain}$$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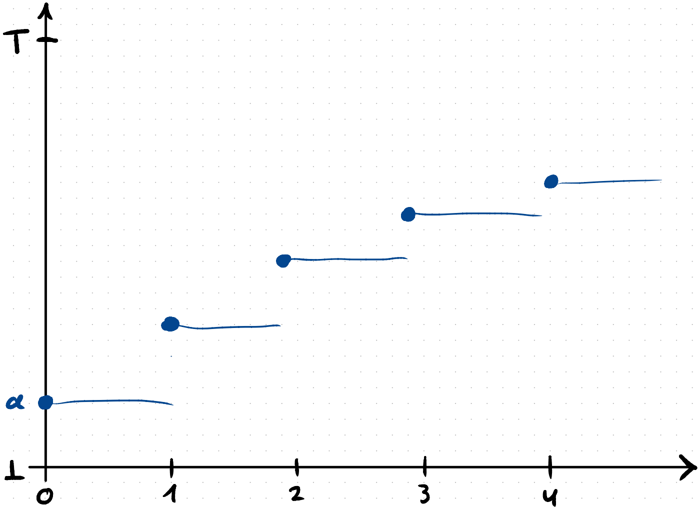
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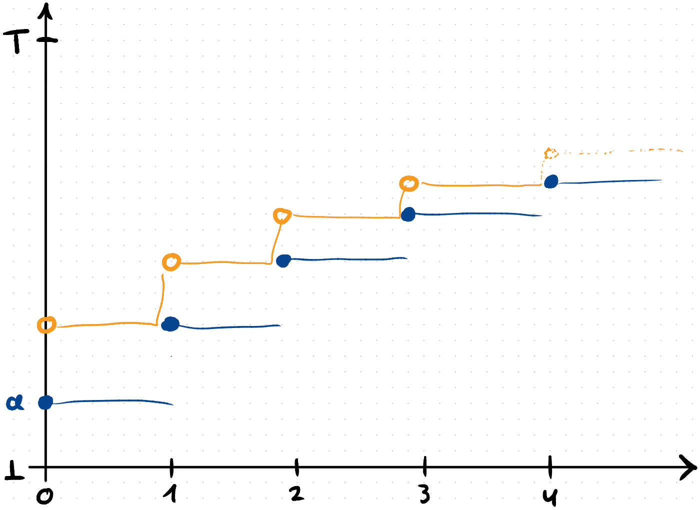
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Interesting Judgements

$a \;\preceq\; \bigcirc\, a$	a is an ascending chain
$\bigcirc\, a \;\preceq\; a$	a is a descending chain
$\Diamond\, a \;\preceq\; \Box\, a$	a is a flat (a constant sequence)
$\Box \Diamond\, a \;\preceq\; \Diamond \Box\, a$	a converges
$F\, a \;\preceq\; a$	all elements of a are prefixed points of F
$a \;\preceq\; F\, a$	all elements of a are postfixed points of F
$F\, a \;\equiv\; a$	all elements of a are fixed points of F
$F \Diamond F^{*} \bot \;\equiv\; \Diamond F^{*} \bot$	all elements of $\Diamond F^{*} \bot$ are fixed points of F

Proofs

Axioms

$$\frac{\text{REFLEX}}{a \;\leq\; a} \qquad \frac{\text{BOT}}{\bot \;\leq\; a} \qquad \frac{\text{TOP}}{a \;\leq\; \top}$$

Inference Rules

$$\frac{a \preceq b}{a \preceq b \curlyvee c}\ \textsc{$\curlyvee$-introR} \qquad \frac{a \preceq c \qquad b \preceq c}{a \curlyvee b \preceq c}\ \textsc{$\curlyvee$-introL}$$

Inference Rules

$$\frac{a \preceq b \qquad b \preceq c}{a \preceq c}\ \textsc{cut}$$

Axioms for Limit Modalities

$$\frac{\Diamond\text{-INFLATE}}{a \preceq \Diamond a} \qquad \frac{\Box\text{-DEFLATE}}{\Box a \preceq a}$$

Induction Principles for $\bigcirc$

$$\frac{\bigcirc a \;\preceq\; a}{\Diamond a \;\preceq\; a}\;\bigcirc\text{-IND} \qquad\qquad \frac{a \;\preceq\; \bigcirc a}{a \;\preceq\; \Box a}\;\bigcirc\text{-COIND}$$

Monotonic Convergence Theorem

"Every monotonic sequence converges."

$$
\dfrac{\dfrac{\dfrac{a \preceq \bigcirc a}{a \preceq \Box a}\ \bigcirc\text{-COIND}}{\Diamond a \preceq \Diamond \Box a}\ \Diamond\text{-MONO}}{\Box \Diamond a \preceq \Diamond \Box a}\ \Box\text{-INTRO}
\qquad
\dfrac{\dfrac{\dfrac{\bigcirc a \preceq a}{\Diamond a \preceq a}\ \bigcirc\text{-IND}}{\Box \Diamond a \preceq \Box a}\ \Box\text{-MONO}}{\Box \Diamond a \preceq \Diamond \Box a}\ \Diamond\text{-INTRO}
$$

Monotonicity and Continuity of Functions

$$\frac{a \;\preceq\; b}{F\,a \;\preceq\; F\,b}\;F\text{-MONO} \qquad \frac{\text{SEMI-CONT}}{\Diamond F\,a \;\preceq\; F \Diamond a}$$

$$\frac{\text{C-CONT}}{F \Diamond a \;\preceq\; \Diamond F\,a} \qquad \frac{a \;\preceq\; \bigcirc a}{F \Diamond a \;\preceq\; \Diamond F\,a}\;\omega\text{-CONT}$$

Inference Rules for Functions

$$\frac{a \;\preceq\; F\,a}{a \;\preceq\; F^{*}a}\;\; F^{*}\text{-}\textsc{introR} \qquad\qquad \frac{F^{*}\text{-}\textsc{iter}}{\bigcirc\, F^{*}a \;\equiv\; F\,F^{*}\bigcirc\, a}$$

Fixed Point Theorems in Iteration Logic

Tarski-Kantorovich Principle

"If $a \preceq F(a)$, then $\sup_k F^k(a)$ is the least fixed point of F above a."

$$\dfrac{\dfrac{a \preceq F\,a}{\vdots} \qquad \dfrac{a \preceq \bigcirc\, a}{\vdots}}{F \Diamond F^* a \preceq \Diamond F^* a} \qquad\qquad \dfrac{\dfrac{a \preceq F\,a}{\vdots}}{\Diamond F^* a \preceq F \Diamond F^* a}$$

$$\dfrac{\dfrac{a \preceq F\,a}{\vdots}}{a \preceq \Diamond F^* a} \qquad\qquad \dfrac{\dfrac{a \preceq b}{\vdots} \qquad \dfrac{F\,b \preceq b}{\vdots}}{\Diamond F^* a \preceq b}$$

(Almost) immediately yields Kleene for $a = \bot$.

A Look into Tarski-Kantorovich Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \;\preceq\; \bigcirc a}\ \textsc{ass.}}{F^* a \;\preceq\; F^* \bigcirc a}\ F^*\text{-}\textsc{mono}}{F\,F^* a \;\preceq\; F\,F^* \bigcirc a}\ F\text{-}\textsc{mono}}{F\,F^* a \;\preceq\; \bigcirc F^* a}\ \textsc{iter}}{\Diamond F\,F^* a \;\preceq\; \Diamond \bigcirc F^* a}\ \Diamond\text{-}\textsc{mono}}{\Diamond F\,F^* a \;\preceq\; F^* a \curlyvee \Diamond \bigcirc F^* a}\ \curlyvee\text{-}\textsc{introR}}{\Diamond F\,F^* a \;\preceq\; \Diamond F^* a}\ \Diamond\text{-}\textsc{exp}
$$

Olszewski Fixed Point Theorem

"$\sup_k \inf_{k \leq j} F^j(a)$ is some fixed point of F."

$$\dfrac{\dfrac{a \preceq \bigcirc a}{\vdots}}{F \Diamond \Box F^* a \preceq \Diamond \Box F^* a} \qquad \dfrac{\dfrac{\bigcirc a \preceq a}{\vdots}}{\Diamond \Box F^* a \preceq F \Diamond \Box F^* a}$$

A Look into Olszewski Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{ASS.}}{\bigcirc\, a \;\preceq\; a}}{F^*\bigcirc\, a \;\preceq\; F^* a}\;F^*\text{-MONO}}{F\,F^*\bigcirc\, a \;\preceq\; F\,F^* a}\;F\text{-MONO}}{\bigcirc\, F^* a \;\preceq\; F\,F^* a}\;\text{ITER}}{\Diamond\bigcirc\, F^* a \;\preceq\; \Diamond F\,F^* a}\;\Diamond\text{-MONO}}{\Box\Diamond\bigcirc\, F^* a \;\preceq\; \Box\Diamond F\,F^* a}\;\Box\text{-MONO}}{\Box\bigcirc\Diamond F^* a \;\preceq\; \Box\Diamond F\,F^* a}\;\bigcirc\Diamond\text{-COMM}}{\Diamond F^* a \curlywedge \Box\bigcirc\Diamond F^* a \;\preceq\; \Box\Diamond F\,F^* a}\;\curlywedge\text{-INTROL}}{\Box\Diamond F^* a \;\preceq\; \Box\Diamond F\,F^* a}\;\Box\text{-EXP}$$

Future Work

- Go transfinite
- More fixed point theorems
- Dispose of concrete semantics. Go fully axiomatic
  - What are other “models” of iteration logic beyond sequences?
- Incorporate Löb Induction
- Can we interpret $\Diamond \Box F^{*} a$ *temporally* in the context of verification?