Monotone Program Analysis

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Outline

Introduction

Background and Preliminaries

A Framework for Compositional and Monotone Termination Analysis

A Weak Theory of Nonlinear Arithmetics with Applications

Experimental Evaluation

Takeaways

Ensuring correctness of programs

Incorrect software has cost time, money, and human lives.[1]

- Non-terminating device driver code leads to non-responsive systems
- Software errors in the Ariane 5 rocket cost about $370M
- Therac-25 software errors caused deaths of cancer patients
- ...

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[1] `https://www.cs.tau.ac.il/~nachumd/horror.html`

Why can't we solve the problem once and for all?

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There does not exist an algorithm that solves the halting problem for every program and input.

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Thus

- Automated verification usually involves trade-offs between soundness, precision, resource consumption, etc.;
- Studying programs and loops with restricted forms is justified;
- Use of heuristics that only work *sometimes* can be justified.

A troubled user of our termination provers

	CPAChecker	UAutomizer	2LS

A troubled user of our termination provers

```
int n = 4096;
for (int i = 0; i < n; i++)
  for (int j = 0; j < n; j++)
    ; // skip
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✗	✓	✓

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The unpredictability problem

Changes in the source program may have unpredictable effects on the analysis results.

The undesirable and the desirable

Program verification is currently unusable, because current tools require too much expertise from the user. Stated differently, tools can understand our programs, but we cannot understand our tools.
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We want to make verification part of the continuous integration pipeline...
– Peter

A troubled (expert) user

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lemma SquareRoot2NotRational(p: nat, q: nat)
requires p > 0 && q > 0
ensures p * p !=  2 * q * q
{
if (p * p) ==  2 * (q * q) {
    calc == {
        (2 * q - p) * (2 * q - p);
        4 * q * q + p * p - 4 * p * q;
        {assert 2 * q * q == p * p;}
        2 * q * q + 2 * p * p - 4 * p * q;
        2 * (p - q) * (p - q);
    }
    SquareRoot2NotRational(2 * q - p, p - q);
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}
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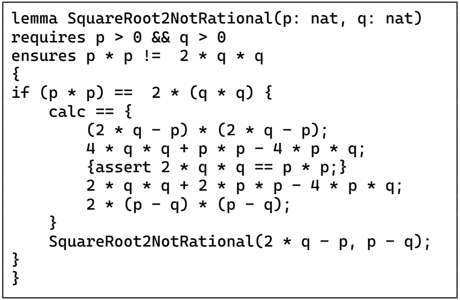

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A troubled (expert) user

This story from the Github issue[1] illustrates

1. Users want behaviors of tools to be somehow predictable.
2. Nonlinear verification conditions can cause unpredictable behavior.

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We consider one property that makes the behavior of tools more predictable:

Monotonicity

A monotone analysis improves or maintains the same result when given a problem that is “not more difficult” than the original.

Proposition: tool users will appreciate such properties!

Can you make it more concrete?

Example 1: reducing reachable states

- Before: a tool can prove a program P terminates unconditionally.
- After: can the tool prove that P terminates under precondition $x \geq 0$?

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Example 2: adding loop invariants

- Before: a tool can prove an assertion C in a program P.
- After: a user annotates P with an additional loop invariant I. Can the tool still prove the assertion?

Sources of non-monotonicity

1. Widening and narrowing in abstract interpretation.
2. Heuristics in model checking used by abstraction/refinement methods, e.g., selecting predicates from counterexamples.
3. Techniques based on syntactic features of code rather than semantics.
4. Undecidability of nonlinear integer arithmetic (NIA).
5. ...

My Research

I have worked on predictable analyses. In particular, I present:

- A framework for compositional and monotone termination analysis.
- A weak theory of nonlinear arithmetic that enables monotone nonlinear invariant generation and synthesis of polynomial ranking functions.

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Fix finite sets of pre-state program variables X and corresponding post-state variables X'.

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- A state formula $S(X)$ is a formula with free variables drawn from X.
- A transition formula $F(X, X')$ is formula with free variables drawn from $X \cup X'$.
- Transition formulas characterize semantics of programs
    - statement `i := i + 1`
    - formula $F \triangleq i' = i + 1 \land j' = j$
    - transition system $(\mathbb{Q}^2, \xrightarrow{F})$ with a transition relation on states $\{i \mapsto x, j \mapsto y\} \rightarrow_F \{i \mapsto x + 1, j \mapsto y\}$.

Regular and ω-regular expressions

Let Σ be an alphabet. Define syntax for labels, regular expressions (Kleene), and ω-regular expressions (Büchi) as

$$\begin{aligned} a &\in \Sigma \\ e \in \mathsf{RegExp}(\Sigma) &::= a \mid 0 \mid 1 \mid e_1 + e_2 \mid e_1\, e_2 \mid e^* \\ f \in \omega\text{-}\mathsf{RegExp}(\Sigma) &::= e^\omega \mid ef \mid f_1 + f_2 \end{aligned}$$

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\end{aligned}
$$

- A regular expression recognizes a set of finite strings.
- An ω-regular expression recognizes a set of *infinite* strings.
  - A^ω recognize all words obtained by concatenating words from A infinitely many times, e.g., $(b^*\,a)^\omega$ recognizes all words with infinitely many a's.

Algebraic program analysis (1)

Algebraic program analysis [Tar81b, FK15] is a form of abstract interpretation in that it computes summaries that over-approximate loop dynamics.

- Traditional (iterative) abstract interpretation:
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Convenient to build monotone analyses: APA avoids fixpoint computations in abstract domains thus avoids widening or narrowing.

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The interpretation over-approximates semantics of all finite paths through the program.

Example: APA Step 1

Consider program `while (j < 4096) j++`.

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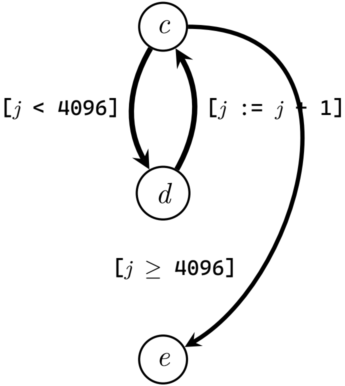
Consider program **`while (j < 4096) j++`**.
Its control flow graph is



A path expression representing all paths is $(\langle c, d\rangle \cdot \langle d, c\rangle)^{\star} \cdot \langle c, e\rangle$.

Example: APA Step 2

We focus on interpreting the part of the expression that represents the loop $(\langle c, d\rangle \cdot \langle d, c\rangle)^\star$:

$\langle c, d\rangle$

`[j < 4096]`

$j < 4096 \wedge j' = j \wedge i' = i$

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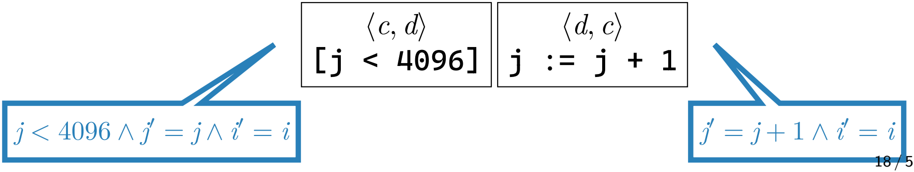

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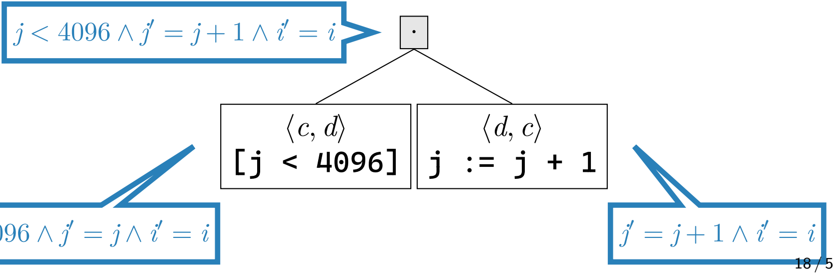

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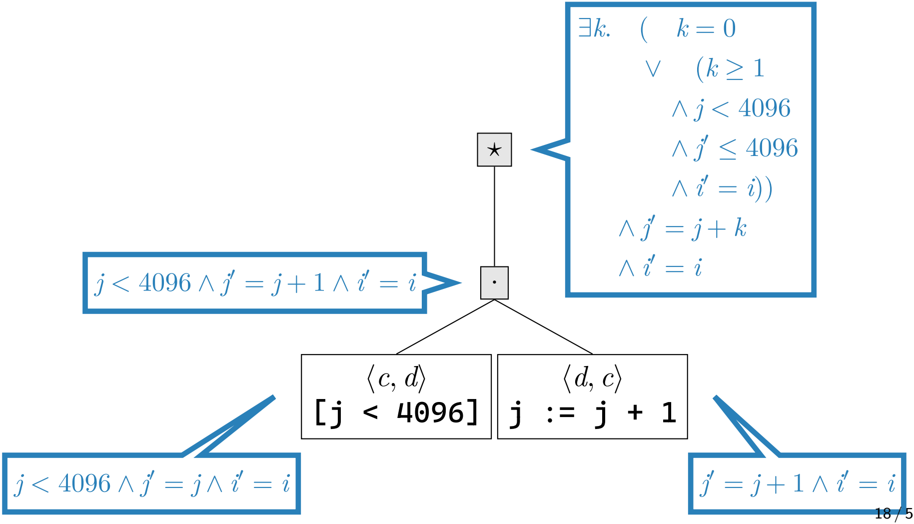

$$\begin{aligned}\exists k.\quad (\quad & k = 0\\ \vee\quad & (k \geq 1\\ & \quad \wedge j < 4096\\ & \quad \wedge j' \leq 4096\\ & \quad \wedge i' = i))\\ \wedge\, & j' = j + k\\ \wedge\, & i' = i\end{aligned}$$

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Monotone recurrences through convex hulls

Define the convex hull of an linear integer arithmetic (LIA) formula $F(Y)$, *conv*(F), to be the strongest formula of the form $AY \geq b$ that is entailed by F, where A is an integer matrix and b is an integer vector. There exist practical algorithms for this [FK15].

We can extract recurrences by invoking this procedure.

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We can extract recurrences by invoking this procedure.

1. Compute *conv*$(\exists X, X'.F(X, X') \wedge \bigwedge_{x \in X} \delta_x = x' - x)$.
2. We get the strongest consequence among those with form $A\delta_x \geq b$.
3. This entails that $F \models Ax' \geq Ax + b$ and we get the strongest such formula.

Invariant generation using recurrences

Generating monotone linear loop summaries for APA [ZK21b].

```
while (x >= 0
    && y >= 0) {
  if (*) {
    x--;
  } else {
    y--;
  }
}
```

Invariant generation using recurrences

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1. Get strongest recurrence relations.

$$
\begin{aligned}
F \models \quad & (x' + y') = (x + y) - 1 \\
& \wedge x' \leq x \wedge x' \geq x - 1 \\
& \wedge y' \leq y \wedge y' \geq y - 1
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❷ Solve the symbolic system of recurrences.

$$\begin{aligned} F^\star \triangleq \exists k. k \geq 0 \wedge (x' + y') = (x + y) - k \\ \wedge x' \leq x \wedge x' \geq x - k \\ \wedge y' \leq y \wedge y' \geq y - k \end{aligned}$$

❸ Summarize the behavior of the loop using $F^\star$.

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A framework for *practical*, *compositional* termination analyses with *monotonicity* guarantees that extends the algebraic program analysis framework.

Contributions

- We extend Tarjan's method [Tar81a] to compute path expressions for infinite paths.
- We extend algebraic program analysis to handle infinite paths.
- We present a collection of monotone termination analyses that instantiate the framework.

Recipe comparison

Original algebraic program analysis recipe:

1. Compute regular expressions that represent finite program paths
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    - $\star$ operator – approximate the transitive closure of loops
    - ω operator – obtain terminating conditions for a loop
    - $\cdot$ operator – propagate terminating conditions
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   - $\star$ operator – approximate the transitive closure of loops
   - ω operator – obtain terminating conditions for a loop
   - $\cdot$ operator – propagate terminating conditions
   - $+$ operator – combine terminating conditions for different paths
3. The interpretation is a condition under which the whole program terminates

Method Overview

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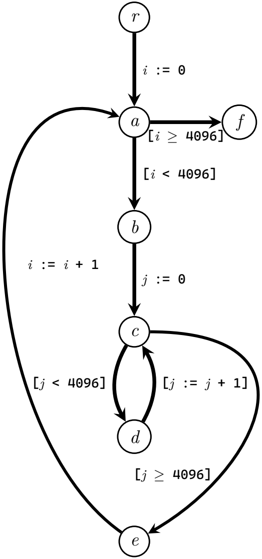

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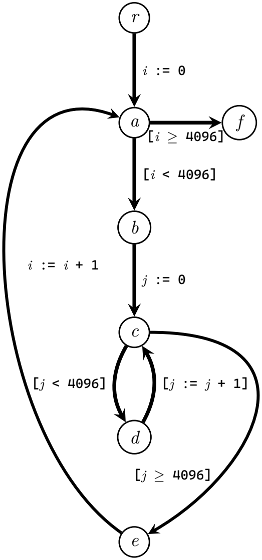
2. Path expression

$$\langle r, a\rangle \left(\begin{array}{l} \overbrace{(\langle a, b\rangle\langle b, c\rangle(\langle c, d\rangle\langle d, c\rangle)^{*}\langle c, e\rangle\langle e, a\rangle)^{\omega}}^{\text{outer loop}} \\ +(\langle a, b\rangle\langle b, c\rangle(\langle c, d\rangle\langle d, c\rangle)^{*}\langle c, e\rangle\langle e, a\rangle)^{*}\langle a, b\rangle\langle b, c\rangle\underbrace{(\langle c, d\rangle\langle d, c\rangle)^{\omega}}_{\text{inner loop}} \end{array} \right)$$



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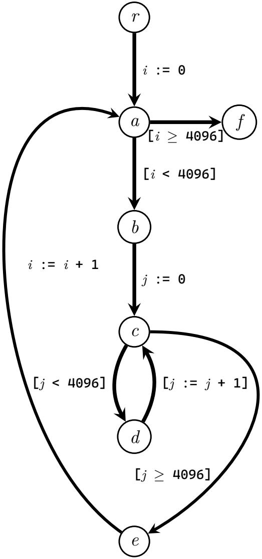

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3. Terminating precondition

true

ω-regular expressions

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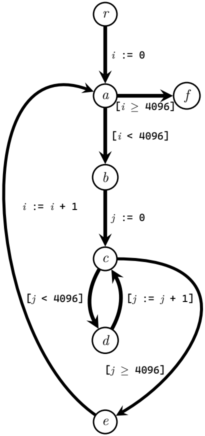

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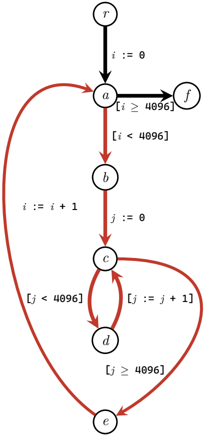
$$\langle r, a\rangle \left(\begin{array}{l} \overbrace{(\langle a, b\rangle\langle b, c\rangle(\langle c, d\rangle\langle d, c\rangle)^*\langle c, e\rangle\langle e, a\rangle)^\omega}^{\text{outer loop}} \\ +(\langle a, b\rangle\langle b, c\rangle(\langle c, d\rangle\langle d, c\rangle)^*\langle c, e\rangle\langle e, a\rangle)^*\langle a, b\rangle\langle b, c\rangle\,\underbrace{(\langle c, d\rangle\langle d, c\rangle)^\omega}_{\text{inner loop}} \end{array} \right)$$



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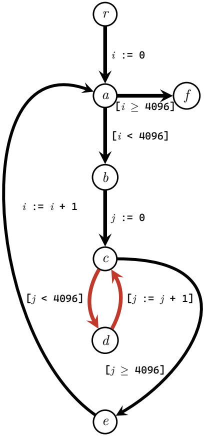

Interpreting ω-regular path expressions

Path expressions are interpreted compositionally in a bottom-up manner.

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composition of relations

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approx. transitive closure [FK15]

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- Analyzing loop summaries makes it decidable
- Other operators are monotone, only need monotone $\star$ and *mp* operators

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Monotonicity

If $F \models G$ then *mp*$(G) \models$ *mp*(F).

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- *mp* combinator based on phase structure: partitions the loop into *phases* and analyzes how program evolves across phases
- mp_{LDS} based on abstracting the loop into a linear dynamical system

mp through linear dynamical system abstraction

Linear loops (linear dynamical systems)

```
while (G(x)) {  // guard is conjunctive
  x = A x;    // matrix multiplication
}
```

Symbolic closed-forms easy to compute, also line of work on decidability of termination of linear loops

- Over the reals [Tiw04]
- Over the rationals [Bra06]
- Over the integers [HOW19]

A monotone *mp* in two steps

Transfer techniques developed for linear loops to reason about termination of general loops:

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- Compute for any loop a *best abstraction* within a particular class of linear dynamical systems.
- Generate terminating conditions for these linear dynamical systems.

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If B terminates starting from states within set X, then $S^{-1}(X)$ leads to termination of A. Suppose A is hard to analyze while B is not, then we may use $S^{-1}(\mathit{mp}(B)) \subseteq \mathit{mp}(A)$ to get some mortal preconditions for A.

Example of a linear simulation

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while (x + y >= 0) {
  if (*) {
    x = x - z;
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For the linear loop on the right, $a' = a - kb$ after k iterations. Thus a mortal precondition is $b > 0$, which implies that a mortal precondition for the original loop is $z = b > 0$.

Which linear abstraction to use?

Q: There are many such linear loops that abstract (i.e., soundly over-approximate) the original loop, which one should we use?

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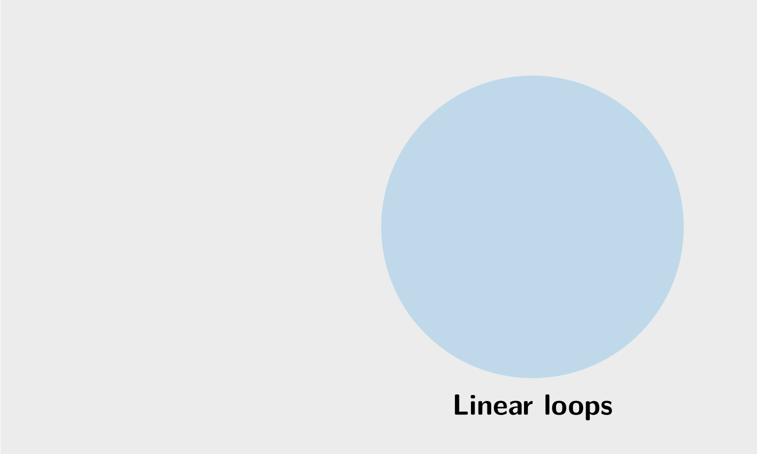
Q: There are many such linear loops that abstract (i.e., soundly over-approximate) the original loop, which one should we use?
A: Use the *best* one which yields the weakest mortal precondition for the original loop (omitting a bunch of linear algebra and category theoretic details here)!

Best abstractions leads to weakest *mp*



Linear loops

Transition systems

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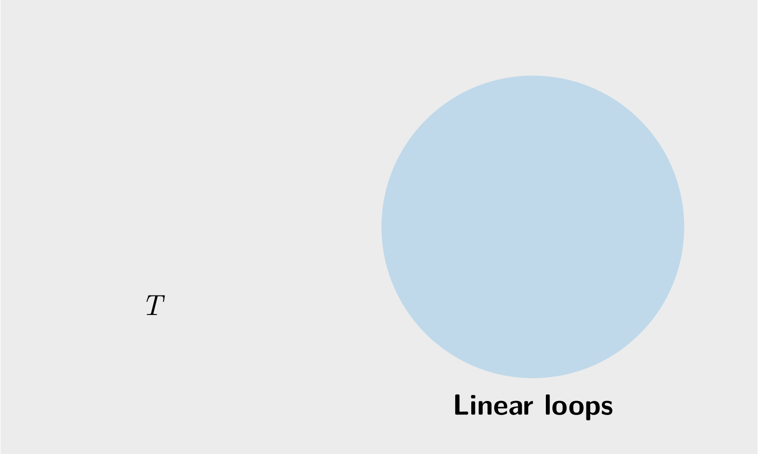

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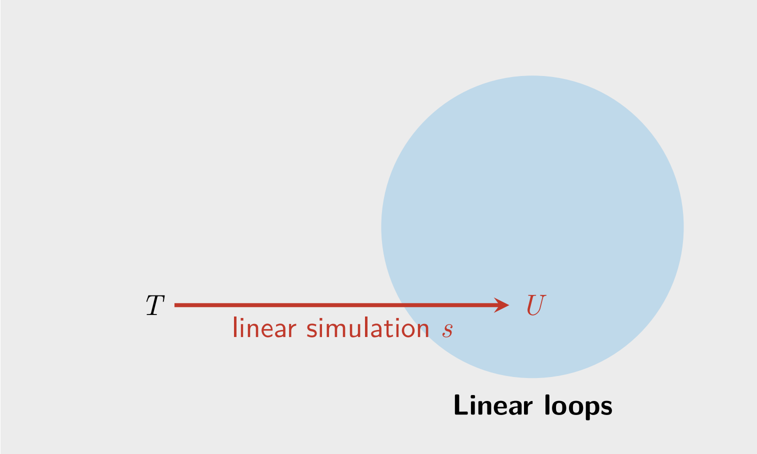

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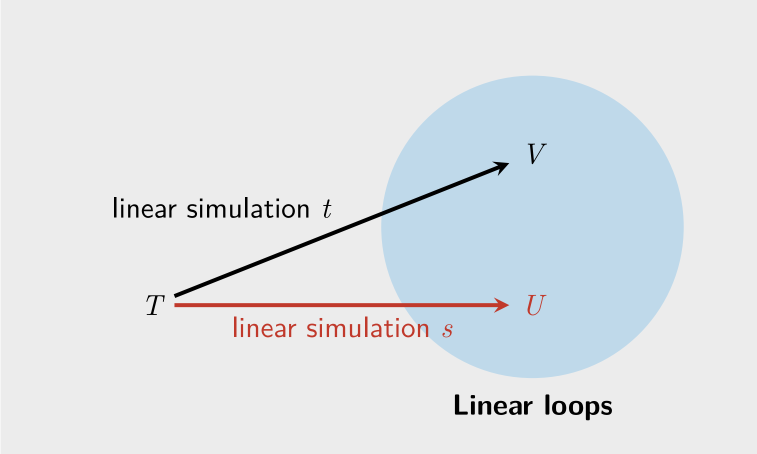

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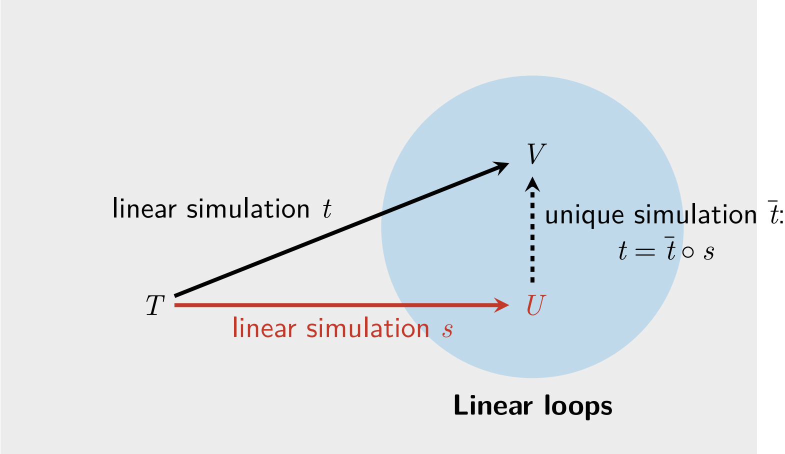

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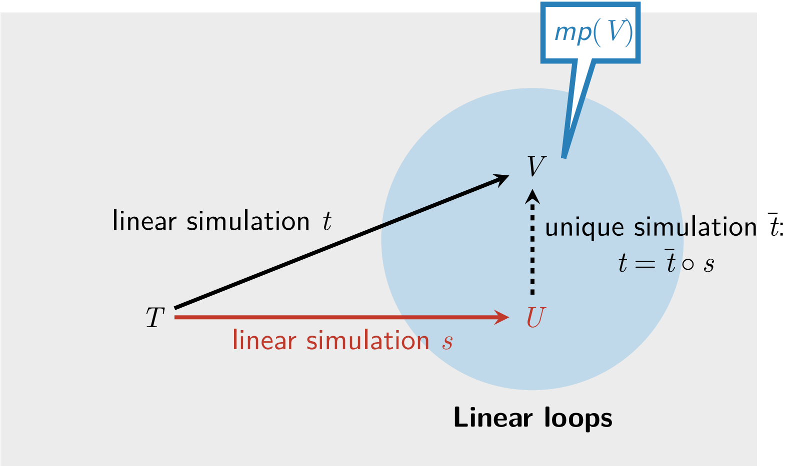

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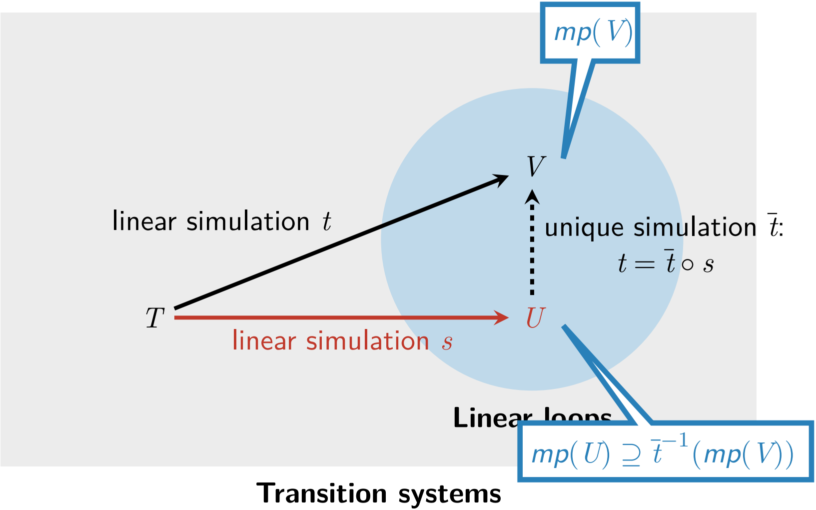

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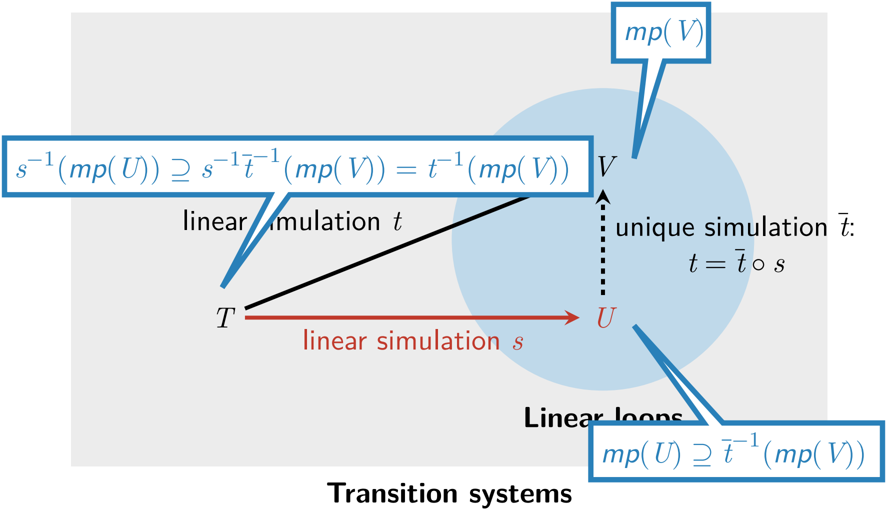

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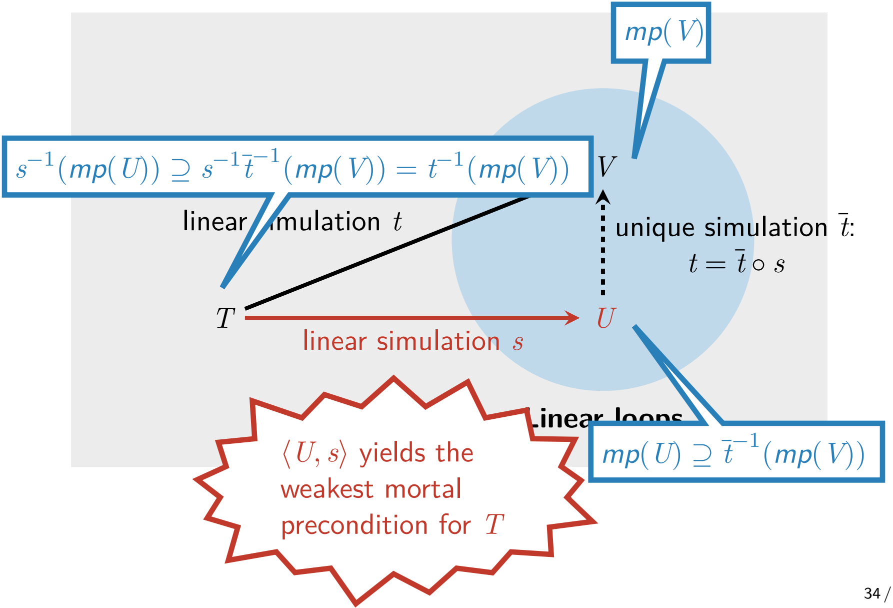

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linear imulation t

unique simulation $\bar{t}$:
$t = \bar{t} \circ s$

linear simulation s

$mp(U) \supseteq \bar{t}^{-1}(mp(V))$

$\langle U, s \rangle$ yields the
weakest mortal
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Key results for linear dynamical system abstraction

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Step II: compute mortal preconditions for linear loops

Given a guard formula $G(\mathbf{x})$ and a linear map $\mathbf{x}' = A\mathbf{x}$ with certain restrictions on A, we can compute mortal preconditions by analyzing the asymptotic behavior of the symbolic closed form $G(A^k\mathbf{x})$.

Example for step II

This method is inspired by Tiwari [Tiw04].

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while (a >= 0) {
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$b < 0 \vee (b = 0 \wedge a \geq 0)$

iff $a^{(k)} \geq 0$ for all k large enough

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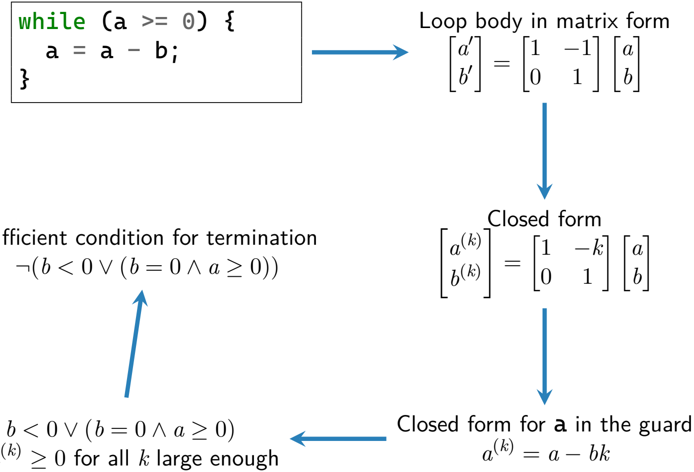

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$$\begin{bmatrix} a^{(k)} \\ b^{(k)} \end{bmatrix} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Closed form for **a** in the guard

$$a^{(k)} = a - bk$$

$$b < 0 \lor (b = 0 \land a \geq 0)$$

iff $a^{(k)} \geq 0$ for all k large enough

Sufficient condition for termination

$$\neg(b < 0 \lor (b = 0 \land a \geq 0))$$

Monotonicity of *mp* based on linear abstraction

Monotonicity of mp_{LDS}

Given transition formulas $F \models G$, the condition we generate for F is weaker than what we generate for G.

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Proof idea: for any transition formula, we have used its best abstraction which yields the weakest possible mortal precondition.

Summary

ComPACT is a practical termination analysis framework, that

- extends algebraic program analysis to handle infinite paths;
- is monotone.

Outline

Introduction

Background and Preliminaries

A Framework for Compositional and Monotone Termination Analysis

A Weak Theory of Nonlinear Arithmetics with Applications

Experimental Evaluation

Takeaways

Linear arithmetic and consequence finding

Linear arithmetic supports *consequence finding*.

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- Extracting linear recurrences: find all linear inequalities implied by a transition formula T with form $F(X, X') \models cx' \leq cx + d$.
- (Complete) linear ranking function synthesis: find all linear terms cx such that $F(X, X') \models (cx' \leq cx - 1 \land cx \geq 0)$.

Nonlinear arithmetic

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Nonlinear integer arithmetic is undecidable, let alone consequence finding!
This work presents:

- A decidable theory of nonlinear arithmetic where strongest consequences of certain forms can be computed.
- A scheme for generating nonlinear loop invariants that are monotone with respect to the proposed theory.
- A scheme for synthesizing nonlinear ranking functions that provides a monotone mortal precondition operator for nonlinear loops.

Intuition for generalizing linear invariants

- Invariants based on linear consequences
  1. Extract a system of linear recurrences entailed by the loop. Specifically, linear terms whose changes are bounded by constants.

$$F(X, X') \models \bigwedge_i t_i x' \leq t_i x + c_i$$

  2. Compute the closed form as a loop invariant.

$$F^\star \triangleq \exists k. k \geq 0 \wedge \bigwedge_i t_i x' \leq t_i x + k c_i$$

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Example

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while (*) {
  x = x + z;
  y = y + z;
  t = x - y;
  z = z + t * t;
  if (*)
    w = w + 1;
  else
    w = w + 2;
}
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Example

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④ The closed form solution approximates loop behavior:

$$\begin{aligned} F^\star \triangleq \exists k. k \geq 0 \wedge & w + k \leq w' \leq w + 2k \\ & \wedge x' - y' = x - y \\ & \wedge z' = z + k(x - y)^2 \end{aligned}$$

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Generating nonlinear invariants

Given transition formula $F(X, X')$ of a loop body, if we can do consequence findings of certain kinds, then we can implement an intuitive recipe:

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3. Extract all recurrences of the form $t_i x' - t_i x - p_i \in \mathbf{C}(F)$, where p_i's are invariant polynomials "generated" by invariant linear term c's.
4. Abstract the dynamics of the loop with nonlinear invariant

$$F^{\star} \triangleq \exists k.\mathit{Int}(k) \wedge k \geq 0 \wedge \bigwedge_i t_i x' \leq t_i x + k p_i \ .$$

Synthesizing nonlinear ranking functions

Given transition formula $F(X, X')$ of a loop body, we could also use consequence finding to prove termination.

mp_{PRF} through consequence finding

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1. Compute the complete set of polynomials that are bounded from below by F:
$$\textit{Bounded}(F) \triangleq \{q : F \models q \geq 0\} .$$
2. Given the previous set, find the subset of polynomials that are also entailed to be decreasing by F:
$$\textit{PRF}(F) \triangleq \{q : q \in \textit{Bounded}(F) \wedge F \models q' \leq q - 1\} .$$

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3. Return true if *PRF*(F) is not empty since F has a polynomial ranking function.

The theory of linear integer/real rings **LIRR**

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- only defines a weakly axiomatized multiplication symbol that generates fewer consequences than the standard model, e.g., it cannot derive $0 \leq xy$ from $0 \leq x \wedge 0 \leq y$.
- admitting methods for manipulating polynomial ideals (for equations) and polyhedral cones (for inequalities).

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Monotonicity of the $\star$ operator

If F, G are transition formulas and $F \models_{\textsf{\textbf{LIRR}}} G$, then $F^\star \models_{\textsf{\textbf{LIRR}}} G^\star$.

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Proof intuition: We have computed all polynomial terms that are implied by F to be bounded from below and decreasing in **LIRR**. Thus the RF synthesis procedure is complete and monotone (in **LIRR**).

Summary

Nonlinear reasoning through **LIRR**

- We can compute strongest consequences of certain forms modulo **LIRR**.
- The complete consequence finding makes it possible to have monotone decision procedures for both safety and liveness properties that require nonlinear reasoning.

Outline

Introduction

Background and Preliminaries

A Framework for Compositional and Monotone Termination Analysis

A Weak Theory of Nonlinear Arithmetics with Applications

Experimental Evaluation

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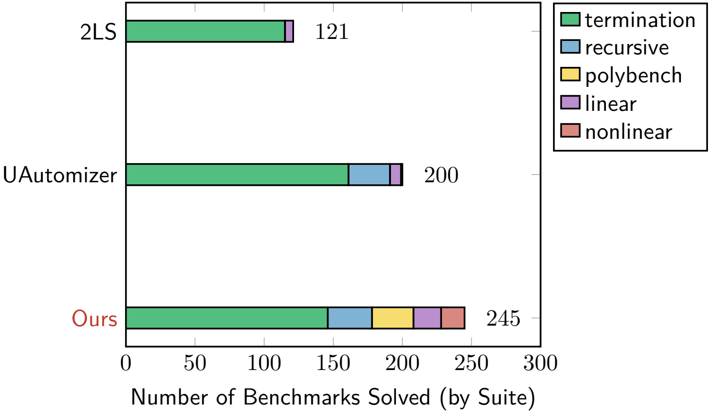
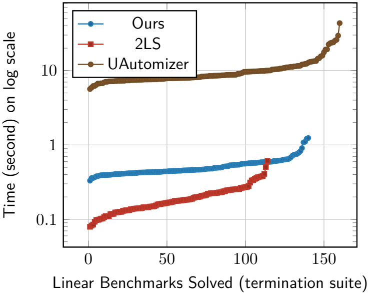
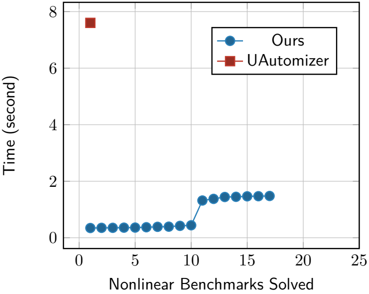
Comparing termination analyses: #benchmarks solved

- Suites termination, recursive: linear programs from SV-COMP/Termination
- Suite polybench: real-world numerical C programs with 10K LOC
- Suite linear: integer linear loops with mod and div
- Suite nonlinear: nonlinear programs from SV-COMP/Termination



Comparing termination analyses: running time





Linear and nonlinear invariant generation

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- Chilon+refine: our invariants + control-flow refinement [CBKR19]
- CRA: monotone invariants by solving linear recurrences [FK15]
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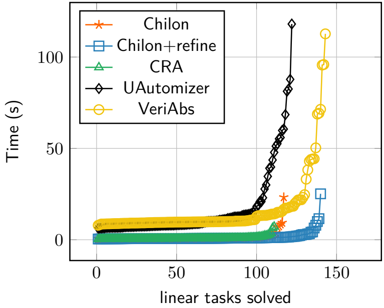
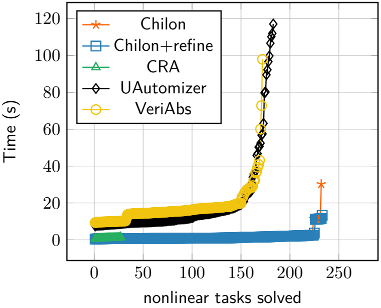




Outline

Introduction

Background and Preliminaries

A Framework for Compositional and Monotone Termination Analysis

A Weak Theory of Nonlinear Arithmetics with Applications

Experimental Evaluation

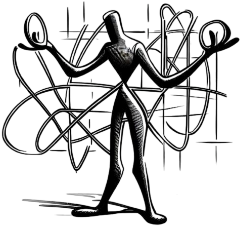
Takeaways

Conclusion

My research shows it is possible to design monotone program analyses that are competitive with state-of-the-art tools in terms of capability and running speed. In particular, I have presented

- A framework for monotone termination analysis [ZK21b, ZK21a].
- A weak theory of nonlinear arithmetic that enables monotone invariant generation [KKZ23] and ranking function synthesis [ZK24].

Questions?



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